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RESPONSE OF A SHAFT TO TRANSVERSE SHOCK

by

FRANCIS H. MOLIN

SUBMITTED IN PARTIAL FULFILLMENT OF THE

REQUIREMENTS FOR THE MASTER OF SCIENCE DEGREE IN

NAVAL ARCHITECTURE AND MARINE ENGINEERING

and

FOR THE PROFESSIONAL DEGREE, NAVAL ENGINEER

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

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PROFESSOR F. M. LEWIS

THESIS SUPERVISOR

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Department of Naval Architecture and Marine
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RESPONSE OF A SHAFT TO TRANSVERSE SHOCK

by

FRANCIS H. MOLIN

Submitted to the Department of Naval Architecture and Marine Engineering on 19 May 1962 in partial fulfillment of the requirements for the Master of Science degree in Naval Architecture and Marine Engineering and the Professional degree, Naval Engineer.

This work was done in part at the M.I.T. Computation Center, Cambridge, Massachusetts.

ABSTRACT

The object of this report is to investigate the response of a shaft or beam to shock applied transversely at the end supports. As a secondary objective, a simplified method of designing a shaft for shock resistance is suggested. In particular, the results of this work demonstrate the prediction of the transient history, with respect to time and position, of the beam deflection, shear force, and bending moment and show a calculation of the times, positions, and maximum amplitudes of these quantities.

To accomplish the objective, velocity shock was assumed and the differential equation for flexural vibration of Timoshenko was solved. The solution was then placed in a form such that values depend only on the ratio of the beam length to radius of gyration of a cross section and the ratio of shear modulus, modified by a factor accounting for section shape, to the modulus of elasticity. The maxima for deflection, shear and bending moment were then tabulated for various length to radius of gyration ratios.

As a model, only a solid, circular, steel shaft of six inch radius was used, but the method of solution could be extended to other configurations.

As a result of the investigation, it can be shown that, comparing all simply supported uniform beams of constant ratios of length to radius of gyration of a cross section and of constant ratio of shear modulus to Young's modulus, modified by section shape, the time histories of deflection, shear, and bending moment will differ only in magnitudes, i.e., the histories of any such beam can be scaled from that of another.

It was found that there is a range of length to radius of gyration ratios where rotary inertia of elemental cross sections and shear effects become significant. Further, there are a two fold infinity of natural frequencies, each of which corresponds to an odd numbered deflection mode.

The higher series of frequencies is associated with axial compression and tension due to bending, and the lower series is associated with the shearing effect. It was found that the higher frequency series, for all modes, goes to infinity as the length to radius of gyration ratio goes to infinity and that, as a result, the equations derived reduce to the classical flexural vibration equations.

It is believed that the results obtained in this work could provide a designer concerned about the effects of shock with a useful tool in anticipating design performance due to shock.

There was an indication that the energy contained in a beam subjected to shock cannot be adequately described with considerations of overall deflection. However, this point should be investigated further.

It is recommended that the equations and curves obtained be checked by experiment. It was noted that the equation for shear converged rather slowly; so, even though the curves obtained from computer calculations included terms through the forty-ninth mode, this may not provide enough accuracy. It is further recommended that other types of supports, especially fixed ends and cantilever, be investigated.

Thesis Supervisor: F. M. Lewis

Title: Professor of Marine Engineering, Emeritus

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A = cross sectional area of beam. (in.^{2})

E = modulus of elasticity. (psi)

G' = Shear modulus. (psi)

$G = k'G'$ (psi)

g = acceleration due to gravity (in./sec.^{2})

I - moment of inertia of a cross section about the neutral axis. (in.^{4})

J = mass moment of inertia of a unit length of shaft about the neutral axis. (lb.-in.-sec.^{2}/in.)

k' = factor relating average shear stress to maximum shear stress.

$k_n = \frac{n\pi}{L}$. (in.$^{-1}$)

L = length of beam. (in.)

M = bending moment. (in.-lbs.)

m = mass per unit length (lb.-sec.^{2}/in.^{2})

n = mode number, an integer ($n = 1,3,5,....$)

Q = shear force. (lbs.)

r = radius. (in.)

r' = radius of gyration of a cross section. (in.)

S_s = shear stress. (psi)

T_{1n}, T_{2n} = time period of the nth mode of the lower and the higher series of frequencies, respectively. (sec.)

t = time. (sec.)

v = initial velocity. (in./sec.)

ω_{1n}, ω_{2n} = frequencies of the nth mode corresponding to the lower and the higher series, respectively (radians/sec.)

x = position along shaft, measured from one end. (in.)

y = transverse deflection. (in.)

z = dummy variable $= 1 + \frac{G}{E}\left[1 + \left(\frac{L}{n\pi r'}\right)^2\right]$.

ρ = density $(\frac{\text{lb.-sec.}^2}{\text{in.}} / \text{in.}^3)$.

<u>Subscripts</u>:

n = mode number = 1,3,5,...

b = refers to bending only.

s = refers to shear only.

Primes, except in k' and r', indicate dimensionless values.

ACKNOWLEDGEMENT

The author wishes to express his deeply felt gratitude to his thesis supervisor, Professor F. M. Lewis, without whose guidance and help this work could not have been done.

I. INTRODUCTION

Today, through use of such devices as armor plating, subdivision, and torpedo bulkhead systems, naval ships are well able to remain afloat after being struck by sizeable amounts of explosives. However, a ship is useless if the shock associated with an explosion causes vital machinery to become inoperable. Furthermore, with modern nuclear weapons, depth charges, and mines available, physical contact with the ship is not always possible; and shock effects are then the primary means of incapacitating ships. This is particularly true for submarines. Indeed in any industrial plant or ship where shock or jolting is anticipated, a design must be able to tolerate these effects in order to perform its mission.

Of particular susceptibility to the effects of shock are shafts and beams when their bearings or supports accelerate perpendicular to the shaft axis. In actual situations, such items as piping, condenser shells, rotating shafts, and in some cases plates could fall into the general category of beams. Of interest to the engineer are such problems as the amount of deflection that must be anticipated in order to have sufficient clearance around the shaft, and the magnitude of the stresses involved in order to assure that no plastic deformation occurs. Thus, this investigation attempts to predict these values.

There have been many works on the dynamic flexural behavior of beams. Among the primary of these is that of Timoshenko [1]. However, these works neglect the effects of shear and rotary inertia which becomes significant when the length to radius of gyration ratio becomes small.

To the author's knowledge, there is no simple design method established to account for shock effects.

The investigation was limited to a solid, circular beam, simply supported. The effects of spinning, such as in rotating machinery, on shaft response was considered negligible; and no calculations account for this.

The excitation was considered to be a "velocity shock", i.e. the beam supports experience an instantaneous constant velocity at the incidence of the shock. This is a commonly used way of specifying shock excitation. Use of velocity shock allows calculations to be simplified; for when a body, initially at rest, is excited by a velocity shock, its response relative to the point of excitation is precisely the same as if the body were initially in motion with a velocity equal to the excitation velocity, and suddenly stopped [2]. Thus calculations can be made as though the beam supports are stationary and an initial constant velocity exists along the length of the beam.

If one looks at a beam subjected to shock in actual practice, the response would depend upon the direction from which the excitation eminated. Thus, if one support were excited before the other, the response would be different than if both ends were excited at the same time. In an attempt to simplify difficult calculations, this report is limited to the case of symmetrical excitation.

Linearity has been assumed in utilizing the equations involved in this report. Thus, the results are limited to beams whose stresses always remain in the elastic region. It was felt that this would not place a serious handicap on a designer if it is assumed that a beam has failed when the stresses resulting from calculations demonstrated herein exceed the yield limit of the material involved.

To sum up, then, the model used for the investigation is a solid circular beam, simply supported, with ends fixed; it is subjected to a velocity shock along its entire length. The response of the model is limited to the elastic limits of the beam material, which was assumed to be steel. The object of the work is to describe the response of the beam model with respect to deflection, shear, and bending moment; then to show to what limits the results apply to other beams than the model chosen; and lastly to propose a simplified method of calculating the deflection and stresses for other beams.

II. PROCEDURE

Definition of Problem. Whereas the physical situation specifies that, at time, $t = o$, the ends attain an instantaneous velocity, v, the problem shall be considered as one where, at $t = o$, the entire length of the beam attains a velocity, -v. The final equation for deflection, then, will be the sum of the values derived and vt.

Figure I indicates the model investigated and the sign conventions. Motion is restricted to the x,y plane. The problem is to find the motion of the beam indicated in Figure I.

Derivation of the Differential Equation of Motion. Assume that the deflection is due to two separate and separable effects, shear and bending. Then total deflection is

$$y = y_s + y_b \ , \tag{1}$$

where

y_s = deflection due to shear only

y_b = deflection due to bending only.

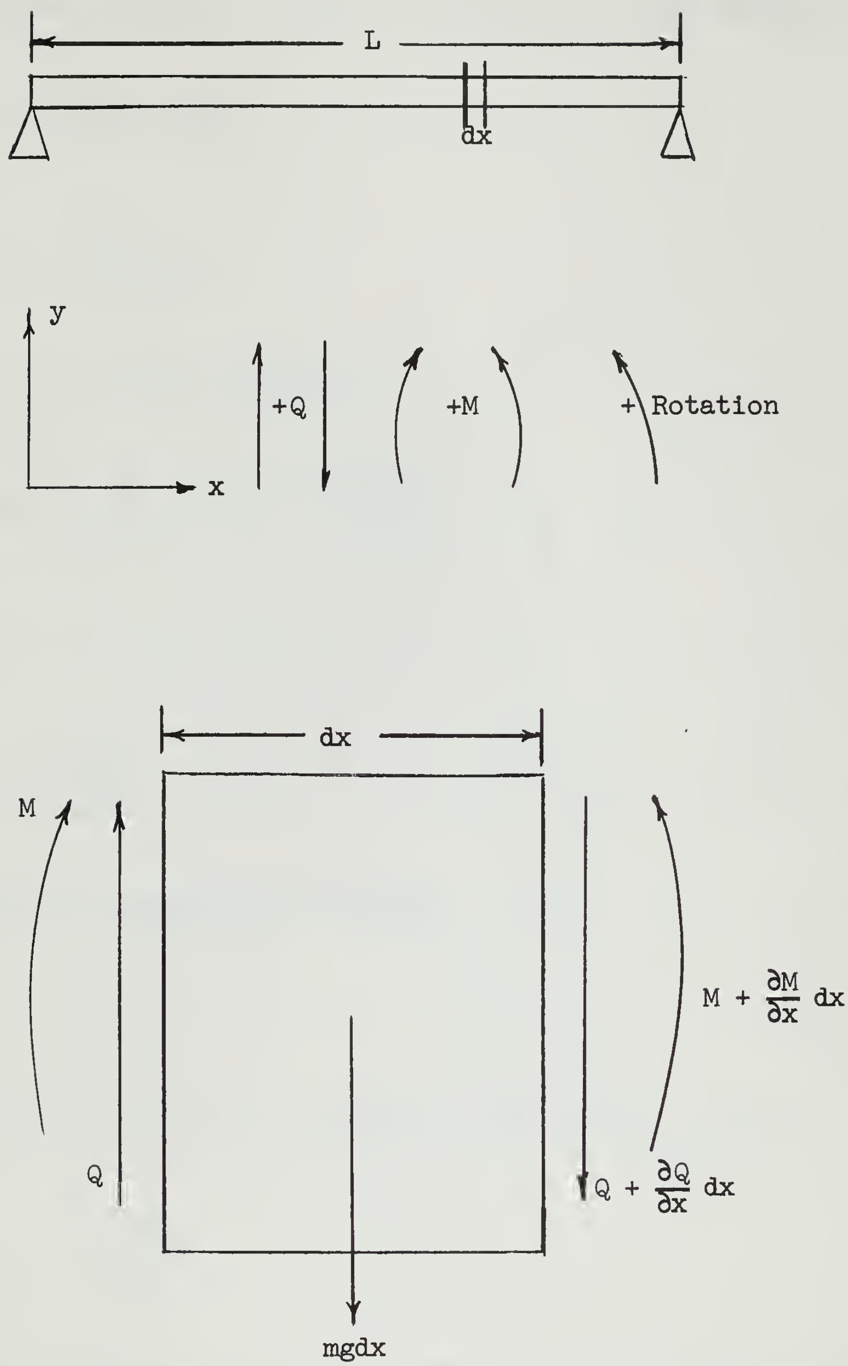
Look now at Figure I. The elemental mass is considered rigid. Consider that, for all shear deflections, there is no rotation of the element. Rotation is associated with bending deflection only. Thus

$$\sum \text{Moments} = Qdx - \frac{\partial M}{\partial x} dx + J\frac{\partial^3 y_b}{\partial t^2 \partial x} dx = 0 \ ,$$

where the last term represents rotary inertia of the elemental mass. Thus it can be seen that, for bending moments, the element always remains

Figure I.

Beam Model and Sign Convention



perpendicular to the slope of the deflection due to bending, $\frac{\partial y_b}{\partial x}$; and $\frac{\partial^3 y_b}{\partial t^2 \partial x}$ simply represents the angular acceleration of the element.

From this it is clear that

$$\frac{\partial M}{\partial x} = Q + J \frac{\partial^3 y_b}{\partial t^2 \partial x} . \tag{2}$$

Equating forces in the y direction,

$$\sum \text{forces} = - \frac{\partial Q}{\partial x} dx - mdx \frac{\partial^2 y}{\partial t^2} = 0 ,$$

or

$$\frac{\partial Q}{\partial x} = - m \frac{\partial^2 y}{\partial t^2} . \tag{3}$$

Recalling that the maximum shear stress,

$$S_{s(max)} = G'\psi ,$$

where ψ represents the slope of deflection due to shear, then

$$S_{s(max)} = -G' \frac{\partial y_s}{\partial x} .$$

Although shear stress is not constant over an entire cross section, it can be shown that the average shear stress, S_s, is

$$S_s = - k' G' \frac{\partial y_s}{\partial x} = - G \frac{\partial y_s}{\partial x} ,$$

where k' is a factor depending upon section shape [1] .

Thus

$$Q = S_s A = - GA \frac{\partial y_s}{\partial x} . \tag{4}$$

Continuing, it is well known that

$$M = + EI \frac{\partial^2 y_b}{\partial x^2} . \tag{5}$$

Combining equations (1), (2), (3), (4), and (5), one obtains

$$EI \frac{\partial^4 y}{\partial x^4} - \left(\frac{EIm}{GA} + J \right) \frac{\partial^4 y}{\partial x^2 \partial t^2} + m \frac{\partial^2 y}{\partial t^2} + \frac{Jm}{GA} \frac{\partial^4 y}{\partial t^4} = 0 . \tag{6}$$

Equation (6) is the equation to be solved. Its derivation is presented in reference [1] , but has been given here in slightly different form because use will be made of the equations involved in the derivation.

Solution of Equation of Motion. Assume that the variables are separable. Then

$$y (x,t) = T (t) X (x) .$$

The equation in x is then of the form

$$A \frac{d^4 X}{dx} + B \frac{d^2 X}{dx} + CX = 0 ,$$

where A, B, and C are functions of time. The solution of equations of this type is of the form

$$X (x) = C_1 \sin k x + C_2 \cos k x + C_3 \sin h k x + C_4 \cos h k x . \tag{7}$$

Since the ends are pinned, deflection and bending moments are always zero at the ends. Thus the boundary conditions are

$$\left.\begin{aligned} X(0) &= X_s(0) + X_b(0) = 0 \\ X(L) &= X_s(L) + X_b(L) = 0 \\ \left.\frac{d^2X_b}{dx^2}\right|_{x=0} &= \left.\frac{d^2X_b}{dx^2}\right|_{x=L} = 0 \end{aligned}\right\} \tag{8}$$

Putting equations (8) into (7) it can be shown that X (x) is of the form,

$$X\ (x) = \sum_{n=1,3,5\ldots}^{\infty} C_n \operatorname{Sin} k_n x\ , \tag{9}$$

where $k_n = \frac{n\pi}{L}$, $n = 1,3,5,\ \ldots.$.

See Appendix A for proof of this.

Assuming no damping, a solution for T (t) can be of the form,

$$T_n\ (t) = C_{1n} \cos \omega_n t + C_{2n} \operatorname{Sin} \omega_n t\ ,$$

or

$$y(x,t) = \sum_{n=1,3,5,\ldots}^{\infty} \left[C_{1n} \cos \omega_n t + C_{2n} \operatorname{Sin} \omega_n t\right] \sin k_n x\ . \tag{10}$$

Putting (10) into (6) it is easily seen that the frequency equation results, i.e.,

$$\frac{Jm}{GA}\ \omega_n^4 - \left[m + \left(\frac{EIm}{GA} + J\right) k_n^2\right] \omega_n^2 + EI\ k_n^4 = 0\ . \tag{11}$$

Equation (11) shows that there are two frequencies associated with each k_n, which fact will be amplified in a later section. Thus, the assumed T(t) is incomplete, since it does not account for both frequencies. Allowing for both frequencies, let

$$y(x,t) = \sum_{n=1,3,5,\ldots}^{\infty} \left[C_{1n} \sin \omega_{1n} t + C_{2n} \cos \omega_{1n} t + C_{3n} \sin \omega_{2n} t + C_{4n} \cos \omega_{2n} t \right] \sin k_n x \, . \tag{12}$$

From equation (12), it is seen that four initial conditions must be utilized to evaluate the constants. Thus, initially, the deflection of the beam is zero. From the definition of the problem, the initial velocity along the entire beam is the velocity of shock, -v . Lastly, due to rotary inertia, the slope and angular velocity of the deflection caused by bending cannot instantaneously change, and are thus zero [3] . Putting this in mathematical terms,

$$y\,(x,0) = 0 \tag{13}$$

$$\left.\frac{\partial y}{\partial t}\right|_{t=o} = -v \tag{14}$$

$$\left.\frac{\partial y_b}{\partial x}\right|_{t=o} = 0 \tag{15}$$

$$\left.\frac{\partial^2 y_b}{\partial x \partial t}\right|_{t=o} = 0 \tag{16}$$

Now, put equation (12) in (3) and integrate. Then

$$Q = \int \frac{\partial Q}{\partial x}\, dx = -m \int \frac{\partial^2 y}{\partial t^2}\, dx = - \sum_{n=1,3,5,\ldots}^{\infty} \frac{m}{kn} \Big[\omega_{1n}^2 C_{1n} \sin \omega_{1n} t$$

$$+ \omega_{1n}^2 C_{2n} \cos \omega_{1n} t + \omega_{2n}^2 C_{3n} \sin \omega_{2n} t + \omega_{2n}^2 C_{4n} \cos \omega_{2n} t \Big] \cos k_n x$$

$$+ \text{Const.} \tag{17}$$

But from Appendix A, it can be seen that, at $x = \frac{L}{2}$, $Q = 0$, thus const. = 0. Apply equation (4) to (17), so that

$$y_s = \int \frac{\partial y_s}{\partial x}\, dx = - \frac{1}{GA} \int Q dx = \sum_{n=1,3,\ldots}^{\infty} \frac{m}{GAk_n^2} \Big[\omega_{1n}^2 C_{1n} \sin \omega_{1n} t$$

$$+ \omega_{1n}^2 C_{2n} \cos \omega_{1n} t + \omega_{2n}^2 C_{3n} \sin \omega_{2n} t + \omega_{2n}^2 C_{4n} \cos \omega_{2n} t \Big] \sin k_n x. \tag{18}$$

Again, boundary conditions preclude constants of integration. Thus, from equations (1), (12), and (18),

$$y_b = \sum_{n=1,3,5,\ldots}^{\infty} \left\{ \left[1 - \frac{m\omega_{1n}^2}{GAk_n^2} \right] \left[C_{1n} \sin \omega_{1n} t + C_{2n} \cos \omega_{1n} t \right] \right.$$

$$\left. + \left[1 - \frac{m w_{2n}^2}{GAk_n^2} \right] \left[C_{3n} \sin \omega_{2n} t + C_{4n} \cos \omega_{2n} t \right] \right\} \sin k_n x. \tag{19}$$

Now the initial conditions can be applied. Applying equation (13) and looking at the nth term,

$$y_n\,(x,0) = (C_{2n} + C_{4n})\sin k_n x = 0,$$

or $$C_{2n} = -C_{4n}\,. \tag{20}$$

Applying equation (14) ,

$$\left.\frac{\partial y_n}{\partial t}\right|_{t=o} = (C_{1n}\,\omega_{1n} + C_{3n}\,\omega_{2n})\sin kx = -v\,,$$

or $$(C_{1n}\,\omega_{1n} + C_{3n}\,\omega_{2n}) = \frac{2}{L}\int_o^L (-v)\sin k_n x dx = -\frac{4v}{n\pi}\,. \tag{21}$$

Continuing, from equation (15) ,

$$\left.\frac{\partial y_b}{\partial x}\right|_{t=o} = k_n\left[C_{2n}\left(1 - \frac{m\omega_{1n}^2}{GAk_n^2}\right) + C_{4n}\left(1 - \frac{m\omega_{2n}^2}{GAk_n^2}\right)\right]\cos k_n x = 0.$$

But from equation (20), this means that,

$$C_{2n}\left(\frac{m\omega_{1n}^2}{GAk_n^2} - \frac{m\omega_{2n}^2}{GAk_n^2}\right) = 0\,.$$

Since, as will be seen later in this work, ω_{1n} cannot equal ω_{2n}, a contradiction results, and

$$C_{2n} = -C_{4n} = 0\,. \tag{22}$$

Lastly, apply equation (16), so that,

$$\left.\frac{\partial^2 y_b}{\partial x\partial t}\right|_{t=o} = k_n\left[C_{1n}\,\omega_{1n}\left(1 - \frac{m\omega_{1n}^2}{GAk_n^2}\right) + C_{3n}\,\omega_{2n}\left(1 - \frac{m\omega_{2n}^2}{GAk_n^2}\right)\right]\cos k_n x$$

$$= 0\,. \tag{23}$$

By combining equations (21) and (23),

$$C_{1n} = \frac{4v}{n\pi} \cdot \frac{(GAk_n^2 - m\omega_{2n}^2)}{m\omega_{1n}(\omega_{2n}^2 - \omega_{1n}^2)} \tag{24}$$

$$C_{3n} = - \frac{4v}{n\pi} \cdot \frac{(GAk_n^2 - m\omega_{1n}^2}{m\omega_{2n}(\omega_{2n}^2 - \omega_{1n}^2)} . \tag{25}$$

Thus, by applying equations (22), (24) and (25) to equation (12) and adding the quantity, vt, as indicated in Definition of Problem above, one finally obtains

$$y(x,t) = v \left\{ t - \sum_{n=1,3,5\ldots}^{\infty} \frac{4}{n\pi} \left[\frac{(GAk_n^2 - m\omega_{1n}^2)}{m\omega_{2n}(\omega_{2n}^2 - \omega_{1n}^2)} \sin \omega_{2n} t \right.\right.$$

$$\left. - \frac{(GAk_n^2 - m\omega_{2n}^2)}{m\omega_{1n}(\omega_{2n}^2 - \omega_{1n}^2)} \sin \omega_{1n} t \right] \sin k_n x \quad . \tag{26}$$

Solution of the Frequency Equation. To complete the solution of the differential equation of motion, the two series of frequencies must be evaluated, i.e. equation (11) must be satisfied.

Thus solving equation (11) for ω_n^2, one finds that

$$\omega_n^2 = \frac{\left[m + \left(\frac{EIm}{GA} + J\right) k_n^2\right] \pm \sqrt{\left[m + \left(\frac{EIm}{GA} + J\right) k_n^2\right]^2 - 4\frac{JmEI}{GA} k_n^4}}{2 \frac{Jm}{GA}} \tag{27}$$

Clearly, there are two frequencies associated with each k_n ; thus, there is a two-fold infinity of frequencies, each pair of which is associated

with an odd numbered mode of deflection. Intuitively, since the deflection was broken into components from shear and from bending, one would expect one of these frequency series to arise from the bending and the other from the shear. This point will be demonstrated more rigorously in a later section.

It has been found more convenient to put equation (27) into the form

$$\omega_n^2 = \frac{2GAk_n^2}{m\left\{\left[1 + \frac{G}{E}\left(1 + \left\{\frac{L}{\pi r'}\right\}^2 \frac{1}{n^2}\right)\right] \pm \sqrt{\left[1 + \frac{G}{E}\left(1 + \left\{\frac{L}{\pi r'}\right\}^2 \frac{1}{n^2}\right)\right]^2 - \frac{4G}{E}}\right\}} \tag{27a}$$

or, letting

$$z = 1 + \frac{G}{E}\left(1 + \left\{\frac{L}{\pi r'}\right\}^2 \frac{1}{n^2}\right) ,$$

$$\omega_n^2 = \frac{2GAk_n^2}{m\left[z \pm \sqrt{z^2 - \frac{4G}{E}}\right]} . \tag{27b}$$

Details of this derivation are given in Appendix B.

Calculation of Shear and Bending Moment. Based on the equations derived in the previous sections, the values of shear and bending moment may now be calculated.

Combining equations (22), (24), (25), and (17) the shear force at any section, at any time, is

$$Q = \sum_{n=1,3,5\ldots}^{\infty} \frac{4v}{n\pi} \left[\frac{(GAk_n^{\,2} - m\omega_{1n}^{\,2})}{k_n\,(\omega_{2n}^{\,2} - \omega_{1n}^{\,2})}\, \omega_{2n} \sin \omega_{2n} t \, - \right.$$

$$\left. - \frac{(GAk_n^{\,2} - m\omega_{2n}^{\,2})}{k_n\,(\omega_{2n}^{\,2} - \omega_{1n}^{\,2})}\, \omega_{1n} \sin \omega_{1n} t \right] \cos k_n x \ . \qquad (28)$$

Utilizing equations (2), (19) and (28), it can be shown that,

$$\frac{\partial M}{\partial x} = \sum_{n=1,3,5\ldots}^{\infty} \frac{4v}{n\pi} \left\{ \left[\frac{m\omega_{2n}}{k_n} + J\left(k_n \omega_{2n} - \frac{m\omega_{2n}^{\,3}}{GAk_n}\right)\right] \left[\frac{GAk_n^{\,2} - m\omega_{1n}^{\,2}}{m\,(\omega_{2n}^{\,2} - \omega_{1n}^{\,2})} \right] \sin \omega_{2n} t \right.$$

$$\left. - \left[\frac{m\omega_{1n}}{k_n} + J\left(k_n \omega_{1n} - \frac{m\omega_{1n}^{\,3}}{GAk_n}\right)\right] \left[\frac{GAk_n^{\,2} - m\omega_{2n}^{\,2}}{m\,(\omega_{2n}^{\,2} - \omega_{1n}^{\,2})} \right] \sin \omega_{1n} t \right\} \cos k_n x \ .$$

Integrating this,

$$M = \sum_{n-1,3,5\ldots}^{\infty} \frac{4v}{n\pi k_n} \left\{ \left[\frac{m\omega_{2n}}{k_n} + J\left(k_n \omega_{2n} - \frac{m\omega_{2n}^{\,3}}{GAk_n}\right)\right] \left[\frac{GAk_n^{\,2} - m\omega_{1n}^{\,2}}{m\,(\omega_{2n}^{\,2} - \omega_{1n}^{\,2})} \right] \sin \omega_{2n} t \right.$$

$$\left. - \left[\frac{m\omega_{1n}}{k_n} + J\left(k_n \omega_{1n} - \frac{m\omega_{1n}^{\,3}}{GAk_n}\right)\right] \left[\frac{GAk_n^{\,2} - m\omega_{1n}^{\,2}}{m\,(\omega_{2n}^{\,2} - \omega_{1n}^{\,2})} \right] \sin \omega_1 t \right\} \sin k_n x \, , \qquad (29)$$

where once again boundry conditions preclude a constant of integration.

<u>Evaluation of Equations</u>. An indication of the validity of any equation occurs when the equation reduces to a simpler, classically known form

based on restrictions of the parameters. Accordingly, look at the values of y, Q, and M when L/r' goes to infinity.

Thus, from equation (27a),

$$\lim_{\frac{L}{r'} \to \infty} \omega_n^2 = \frac{2GAk_n^2}{m\left[\frac{G}{E}\left(\frac{L}{\pi r' n}\right)^2 \pm \frac{G}{E}\left(\frac{L}{\pi r' n}\right)^2\right]}$$

from which it can be seen that,

$$\omega_{2n}^2 \longrightarrow \infty ,$$

$$\omega_{1n}^2 \longrightarrow \frac{EAk_n^2 n^2 \pi^2 r'^2}{mL^2} = k_n^4 \frac{EI}{m} ,$$

or

$$\omega_{2n} \longrightarrow \infty ; \quad \omega_{1n} \longrightarrow k_n^2 \sqrt{\frac{EI}{m}} . \tag{30 a,b}$$

Notice that ω_{1n} is the classical natural frequency for beams of long length to radius ratio. Putting equations (30 a,b) into (26) then

$$\lim_{\frac{L}{r'} \to \infty} y(x,t) = v\left\{t - \sum_{n=1,3,5\ldots}^{\infty} \frac{4}{n}\left[\frac{1}{\omega_{1n}} \sin \omega_{1n} t \sin k_n x\right]\right\}$$

or,

$$y(x,t) = v\left\{t - \sum_{n=1,3,5\ldots}^{\infty} \frac{4}{n}\left(\frac{L^2}{n^2 \pi^2}\right)\sqrt{\frac{m}{EI}} \sin k_n^2 \sqrt{\frac{EI}{m}}t \sin k_n x\right\}. \tag{31}$$

But this is just the classical deflection equation for large $\frac{L}{r'}$ [1]

Now, look at the bending moment equation, equation (29). At first glance, it appears that the coefficient of the $\sin \omega_{2n} t$ term blows up

as $\frac{L}{r} \longrightarrow \infty$. However, it can be shown that this term actually goes to zero. This is because the mass moment of inertia, J, times k_n goes to zero. See Appendix C for proof of this convergence.

Appendix C also indicates that

$$\lim_{\frac{L}{r'} \to \infty} M = \sum_{n=1,3,5\ldots}^{\infty} \frac{4v}{n} \frac{m\omega_{1n}}{k_n^2} \sin \omega_{1n} t \sin k_n x \quad ,$$

or, for large $\frac{L}{r'}$,

$$M = \frac{4v}{\pi} \sum_{n=1,3,5\ldots}^{\infty} \frac{\sqrt{EIm}}{n} \sin \ k_n^2 \sqrt{\frac{EI}{m}} t \cdot \sin k_n x \quad . \tag{32}$$

To further check this, it can be seen by inspection that using equation (31) to determine M from the formula,

$$M = E I \frac{\partial^2 y}{\partial x^2} ,$$

equation (32) results.

The shear equation does not converge when $\frac{L}{r'} \longrightarrow \infty$, as is the case when classical theory is utilized. This fact shall not be derived here.

Next, look at the equations for y, Q, and M when $\frac{L}{r'n}$ goes to zero. Notice that this can occur in a beam of small $\frac{L}{r'}$ for any mode, n ; or for any beam at a large n.

Substituting equation (27b) in (26), (28), and (29), it can be shown that

$$y(x,t) = v \Bigg\{ t - \frac{4L}{\pi^2} \sqrt{\frac{m}{GA}} \sum_{n=1,3,5...}^{\infty} \frac{1}{n^2} \frac{G}{E} \Bigg\{ \Bigg(1 - \frac{2}{z + \sqrt{z^2 + \frac{4G}{E}}} \Bigg) \cdots\cdots$$

$$\cdots\cdots \left[\frac{z - \sqrt{z^2 - \frac{4G}{E}}}{2 \quad z^2 - \frac{4G}{E}} \right]^{\frac{1}{2}} \sin \omega_{2n} t - \Bigg(1 - \frac{2}{z - \sqrt{z^2 - \frac{4G}{E}}} \Bigg) \cdots\cdots$$

$$\cdots\cdots \left[\frac{z + \sqrt{z^2 - \frac{4G}{E}}}{2 \quad z^2 - \frac{4G}{E}} \right]^{\frac{1}{2}} \sin \omega_{1n} t \Bigg\} \sin k_n x \, , \tag{33}$$

$$Q(x,t) = \frac{4v}{\pi} \sqrt{mGA} \sum_{n=1,3,5...}^{\infty} \frac{1}{n} \Bigg\{ \Bigg(\frac{z + \sqrt{z^2 - \frac{4G}{E}} - 2}{4} \Bigg) \cdots\cdots$$

$$\cdots\cdots \left[\frac{2 \left(z - \sqrt{z^2 - \frac{4G}{E}} \right)}{z^2 - \frac{4G}{E}} \right]^{\frac{1}{2}} \sin \omega_{2n} t - \Bigg(\frac{z - \sqrt{z^2 - \frac{4G}{E}} - 2}{4} \Bigg) \cdots\cdots$$

$$\cdots\cdots \left[\frac{2 \left(z + \sqrt{z^2 - \frac{4G}{E}} \right)}{z^2 - \frac{4G}{E}} \right]^{\frac{1}{2}} \sin \omega_{1n} t \Bigg\} \cos k_n x \, , \tag{34}$$

$$M(x,t) = \frac{4vL}{\pi^2} \sqrt{mGA} \sum_{n=1,3,5...}^{\infty} \frac{1}{n^2} \Bigg\{ - \left[\frac{\left(z - \sqrt{z^2 - \frac{4G}{E}} \right)}{2 \left(z^2 - \frac{4G}{E} \right)} \right]^{\frac{1}{2}} \sin \omega_{2n} t$$

$$+ \left[\frac{\left(z + \sqrt{z^2 - \frac{4G}{E}} \right)}{2 \left(z^2 - \frac{4G}{E} \right)} \right]^{\frac{1}{2}} \sin \omega_{1n} t \Bigg\} \sin k_n x \, . \tag{35}$$

Certainly these equations look forbidding, but their derivation involves only algebra, even though long and complicated, and shall not be indicated here.

Taking the limits of (33), (34), and (35) as L/r' goes to zero, one finds

$$\lim_{\frac{L}{r'}\to 0} y(x,t) = v\left\{t - \frac{4L}{\pi^2}\sqrt{\frac{m}{GA}}\sum_{n=1,3,5\ldots}^{\infty}\frac{1}{n^2}\sin\omega_{1n}t\,\sin k_n x\right. ,\tag{36}$$

$$\lim_{\frac{L}{r'}\to 0} Q(x,t) = \frac{4v\sqrt{mGA}}{\pi}\sum_{n=1,3,5\ldots}^{\infty}\frac{1}{n^{\cancel{2}}}\sin\omega_{1n}t\,\cos k_n x \quad ,\tag{37}$$

$$\lim_{\frac{L}{r'}\to 0} M(x,t) = \frac{4vL\sqrt{mGA}}{\pi^2}\sum_{n=1,3,5\ldots}^{\infty}\frac{1}{n^2}\left[\frac{-\frac{G}{E}}{\left(1-\frac{G}{E}\right)}\sin\omega_{2n}t\right.$$

$$\left. + \frac{1}{\left(1-\frac{G}{E}\right)}\sin\omega_{1n}t\right]\sin k_n x \quad .\tag{38}$$

The frequencies in the above equations are found by

$$\lim_{\frac{L}{r'}\to 0}\omega_n^2 = \frac{2GAk_n^2}{m\left\{\left(1+\frac{G}{E}\right)\pm\left(1-\frac{G}{E}\right)\right\}} ,$$

from which one sees that

$$\omega_{2n}^2 \to \frac{EAk_n^2}{m} \quad ; \quad \omega_{1n}^2 \to \frac{GAk_n^2}{m} \quad .\tag{39 a,b}$$

One might ask why such complicated equations as (33), (34), and (35) were used to obtain the limits instead of merely substituting (39 a,b)

into (26), (28), and (29). Had this been done, the $\sin \omega_{2n}t$ portion of the moment equation would have gone to zero, yielding a faulty equation.

Equations (39 a,b) indicate that ω_{2n} is associated with the bending of the beam since it contains Young's modulus. Similarly, ω_{1n} is associated with the shearing action. This holds true for all modes in beams with small L/r' and for any beam in the higher modes. One could consider that the discrepancy in the lower modes is caused by the interaction of shear and bending. There is not sufficient time for the modification to occur in the higher modes.

With this interpretation, it appears that the response of a very short beam is almost entirely due to shear, which is completely acceptable intuitively. Only the bending moment shows any effects, then, of rotary inertia, since only it has the $\sin \omega_{2n}t$ term.

A zero-length beam probably has no significance physically, but, when it is recalled that as n gets large, the nth term goes to the form of (36), (37), and (38) for all beams, they become more interesting. Notice equation (37) in particular. This equation can be transformed by

$$\frac{4v}{\pi}\sqrt{mGA}\sum_{n=1,3,5\ldots}^{\infty}\frac{1}{n}\left(\sin \omega_{1n}t \cos k_n x\right) = \frac{4v}{\pi}\sqrt{mGA}\sum_{n=1,3,5\ldots}^{\infty}\frac{1}{n}\ \ldots$$

$$\ldots\left[\frac{1}{2}\left(\sin \omega_{1n}t \cos k_n x + \cos \omega_{1n}t \sin k_n x\right) - \frac{1}{2}\left(\cos \omega_{1n}t \sin k_n x\right.\right.$$

$$\left.\left. - \sin \omega_{1n}t \cos k_n x\right)\right] = \frac{4v}{\pi}\sqrt{mGA}\sum_{n=1,3,5\ldots}^{\infty}\frac{1}{n}\left[\frac{1}{2}\sin k_n\left(x - \sqrt{\frac{GA}{m}}\ t\right)\right.$$

$$\left. - \frac{1}{2}\sin k_n\left(x + \sqrt{\frac{GA}{m}}\ t\right)\right] \quad . \tag{40}$$

This is just the equation for square waves traveling in from the ends. The magnitude of each component is $v \frac{\sqrt{mGA}}{2}$. The component traveling in from x = 0 is initially positive, and the one from x = L is initially negative. The waves travel at a velocity,

$$\text{velocity} = \sqrt{\frac{GA}{m}} = \sqrt{\frac{G}{\rho}}$$

Further notice that, after each component has traveled the length of the beam once, a reflected shear wave of magnitude $v\sqrt{mGA}$ results. Thus, the transient response of a shaft could be approximated by this square wave, and the accuracy improves as $\frac{L}{r'}$ gets small.

Lastly, it is noticed that equations (36), (37), and (38) prove the convergence of the basic equations for y, Q, and M, since, as indicated above the terms of each of the equations represent the limit as n gets large. Thus for the higher modes, y_n and M_n decrease as $\frac{1}{n^2}$ and Q_n decreases as $\frac{1}{n}$.

<u>Concept of an Equivalent Beam.</u> An important characteristic of the equations for deflection, shear, and bending moment can be realized if they are non-dimensionalized.

Accordingly, define non-dimensional values with primes. Then looking at equation (27b), one finds that

$$\omega_n'^2 = \frac{\omega_n^2}{\left(\frac{GA}{mL^2}\right)} = \frac{2 n^2 \pi^2}{z \pm \sqrt{z^2 - \frac{4G}{E}}} \quad . \tag{41}$$

The non-dimensional time is defined such that

$$\omega_n' t' = \omega_n t \quad ,$$

whereby

$$t' = \frac{1}{L} \sqrt{\frac{GA}{m}} \quad t \quad . \tag{42}$$

From the definition of k_n ,

$$x' = \frac{x}{L} \tag{43}$$

Continuing, from equations (33), (34), and (35), it is seen that

$$y' = \frac{y}{\left(vL \sqrt{\frac{m}{GA}} \right)} = t' - \frac{4}{\pi^2} \sum_{n=1,3,5\ldots}^{\infty} \frac{1}{n^2} \frac{G}{E} \left\{ \left(1 - \frac{2}{Z + \sqrt{z^2 - \frac{4G}{E}}} \right) \left[\frac{z - \sqrt{z^2 - \frac{4G}{E}}}{2 \left(z^2 - \frac{4G}{E} \right)} \right]^{\frac{1}{2}} \ldots \right.$$

$$\left. \ldots\ldots \sin \omega'_{2n} \; t' - \left(1 - \frac{2}{z - \sqrt{z^2 - \frac{4G}{E}}} \right) \left[\frac{z + \sqrt{z^2 - \frac{4G}{E}}}{2 \left(z^2 - \frac{4G}{E} \right)} \right]^{\frac{1}{2}} \sin \omega'_{nt} t' \right\} \sin n\pi x' \tag{44}$$

$$Q' = \frac{Q}{v \sqrt{mGA}} = \frac{4}{\pi} \sum_{n=1,3,5\ldots}^{\infty} \frac{1}{n} \left\{ \left(\frac{z + \sqrt{z^2 - \frac{4G}{E}} - 2}{4} \right) \ldots\ldots \right.$$

$$\ldots\ldots \left[\frac{2 \left(z - \sqrt{z^2 - \frac{4G}{E}} \right)}{z^2 - \frac{4G}{E}} \right]^{\frac{1}{2}} \sin \omega'_{2n} \; t' - \left(\frac{z - \sqrt{z^2 - \frac{4G}{E}} - 2}{4} \right) \ldots\ldots$$

$$\left. \ldots\ldots \left[\frac{2 \left(z + \sqrt{z^2 - \frac{4G}{E}} \right)}{z^2 - \frac{4G}{E}} \right]^{\frac{1}{2}} \sin \omega'_{1n} t' \right\} \sin n\pi x' \tag{45}$$

$$M' = \frac{m}{vL \sqrt{mGA}} = \frac{4}{\pi^2} \sum_{n=1,3,5\ldots}^{\infty} \frac{1}{n^2} \left\{ \left[\frac{z - \sqrt{z^2 - \frac{4G}{E}}}{2 \left(z^2 - \frac{4G}{E} \right)} \right]^{\frac{1}{2}} \sin \omega'_{2n} t' \right.$$

$$\left. + \left[\frac{z + \sqrt{z^2 - \frac{4G}{E}}}{2 \left(z^2 - \frac{4G}{E} \right)} \right]^{\frac{1}{2}} \sin \omega'_{1n} t' \right\} \sin n\pi \, x' \tag{46}$$

From these equations, it is clear that the non-dimensional values of frequency, deflection, shear, and bending moment are equal for any non-dimensional time or position, for any two beams if their values of z are the same.

Recalling that

$$z = 1 + \frac{G}{E}\left(1 + \left\{\frac{L}{\pi r'}\right\}^2 \frac{1}{n^2}\right),$$

it is thus necessary that, for the non-dimensional equations of any two beams to be equivalent, the ratios $\frac{G}{E}$ and $\frac{L}{r'}$, for one beam must equal these ratios for the other.

Look at the ratio of the modified shear modulus to Young's modulus. As was indicated earlier,

$$G = k' \, G' \quad ,$$

where k' depends on section shape. Evaluation of k' is complex and beyond the scope of this work; however, for a first approximation k' could be evaluated by the equation,

$$k' = \frac{br'^2}{\int_0^C hdA} \quad , \tag{47}$$

where

b = width of the beam material at the neutral axis,

C = height of the extreme fiber of a section measured from the neutral axis,

h = height of any point in a section measured from the neutral axis. [4]

See reference [5] for the exact evaluation of k'. Use of equation (47) yields results accurate to about four per cent for a solid circular cross section.

Clearly there are many variations in shape that would yield a constant value of $\frac{k'G'}{E}$.

<u>Effect of Damping</u>. Even though the equations derived herein are applicable only in the elastic region, hysteresis damping does occur. If the equations for y, Q, and M are used in the form of equations (26), (28), and (29), respectively, there is a possibility that many of the modes will become additive at some relatively long time. The result would be inordinately large magnitudes for the three quantities of interest.

Applying damping in an analytical way would be difficult to say the least. However, its major effect, i.e. that of reducing amplitude, could be indicated approximately without loss of generality for the formulas.

Accordingly let the deflection energy that is lost each cycle be two per cent of the energy of that cycle [6] . From this, it can be shown that a damping factor for each term would be approximately

$$\text{Damping factor} = e^{-0.00161\,\omega_n t} \tag{48}$$

See Appendix D for details of this derivation.

<u>Suggested Design Method</u>. A method of calculating the response of any beam is implied by the concept of an equivalent beam. Using equations (42), (44), (45), and (46), modified by damping factors, one could

obtain values of maximum shear, deflection and bending moments and the time at which these occur. For a constant $\frac{G}{E}$, these maxima could be plotted versus $\frac{L}{r'}$. Several plots for various $\frac{G}{E}$ would result in a set of curves that would be completely general. After a few simple calculations to convert from the non-dimensional magnitudes to absolute magnitudes, one would have a good indication of the stresses experienced by the shaft of interest.

III. RESULTS

The equations for deflection, shear force and bending moment, modified by the appropriate damping factors, were programmed on the IBM 7090 computer for a solid steel, circular beam. The results of that computation are indicated in Figures II, III, and IV. These plots are for unity excitation velocity; thus one need only multiply the actual velocity to obtain the magnitudes desired.

Figure II shows the time history of y, Q and M for a beam of six inch radius and thirty-six inch length. Notice that, for a unit velocity input, the deflection curve magnitude could be considered in units of time. The origin of each shear and moment curve is the horizontal line representing the position of the end points on the corresponding deflection curve. Since the excitation is symmetrical only one-half of the beam is represented.

Figure III is included to obtain a qualitative picture of the comparison of the responses of two beams of different $\frac{L}{r}$, . The solid line is for the beam of Figure II while the dashed line represents the response of a beam of six inch radius and sixty inch length. The units of this figure have been normalized by dividing time by length and y, Q, and M by the product of velocity and length. Since, for both beams, the cross section is the same, equations (42), (44), (45), and (46) indicate the validity of this normalization. Notice that the magnitudes have not been non-dimensionalized, although that too could have been done. Nevertheless the normalizing method employed provides a satisfactory base for comparison.

Figure IV shows the variation of the magnitude of the maximum deflection, shear, and bending with respect to $\frac{L}{r'}$. Here all units have been non-dimensionalized in accordance with equations (44), (45), and (46). The position of the maximum deflection and bending moment was, as would be expected, the center of the beam. The position of the maximum shear varied from $x' = 0$ to about $x' = .125$.

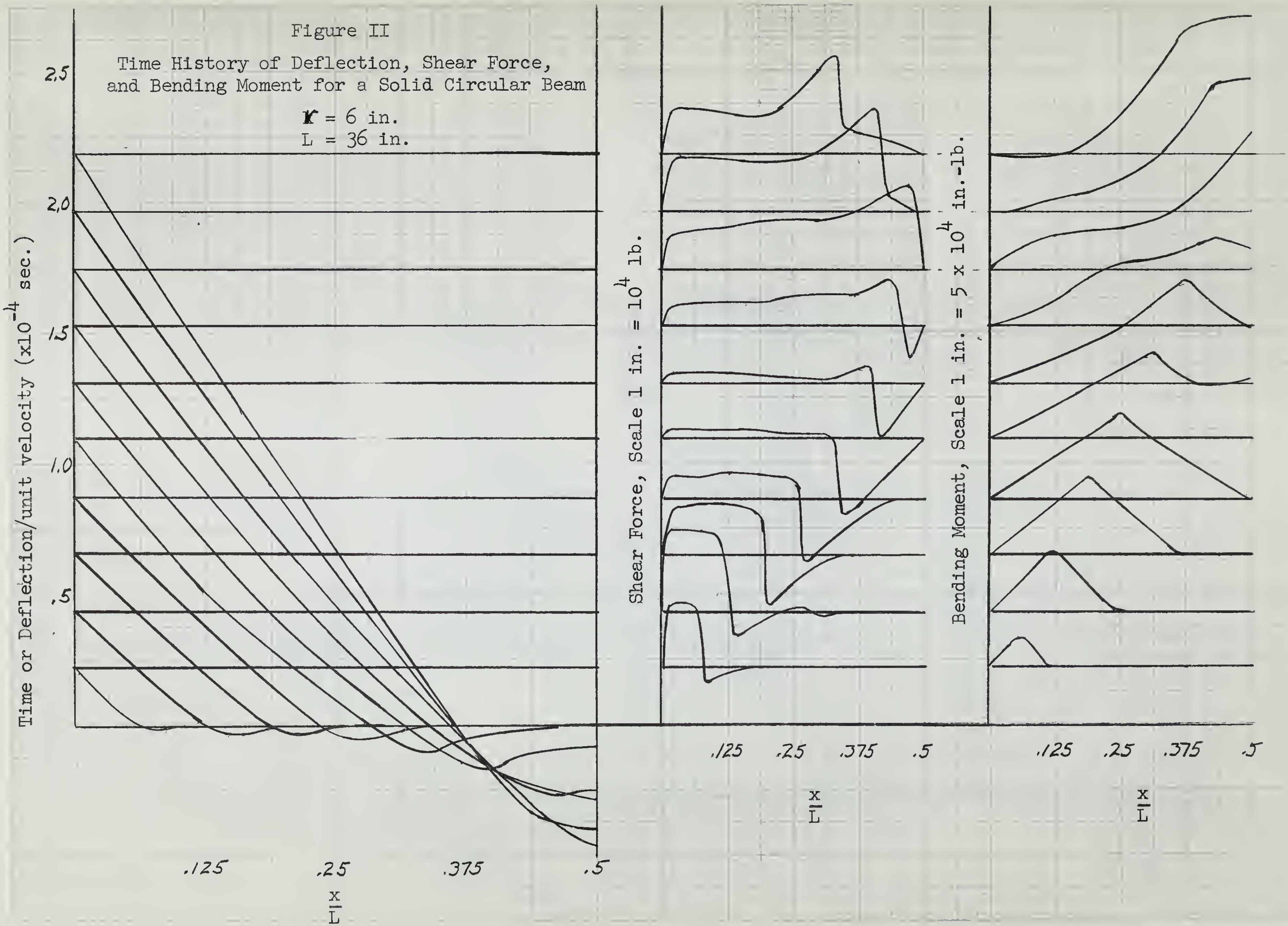


Figure II

Time History of Deflection, Shear Force, and Bending Moment for a Solid Circular Beam

r = 6 in.
L = 36 in.

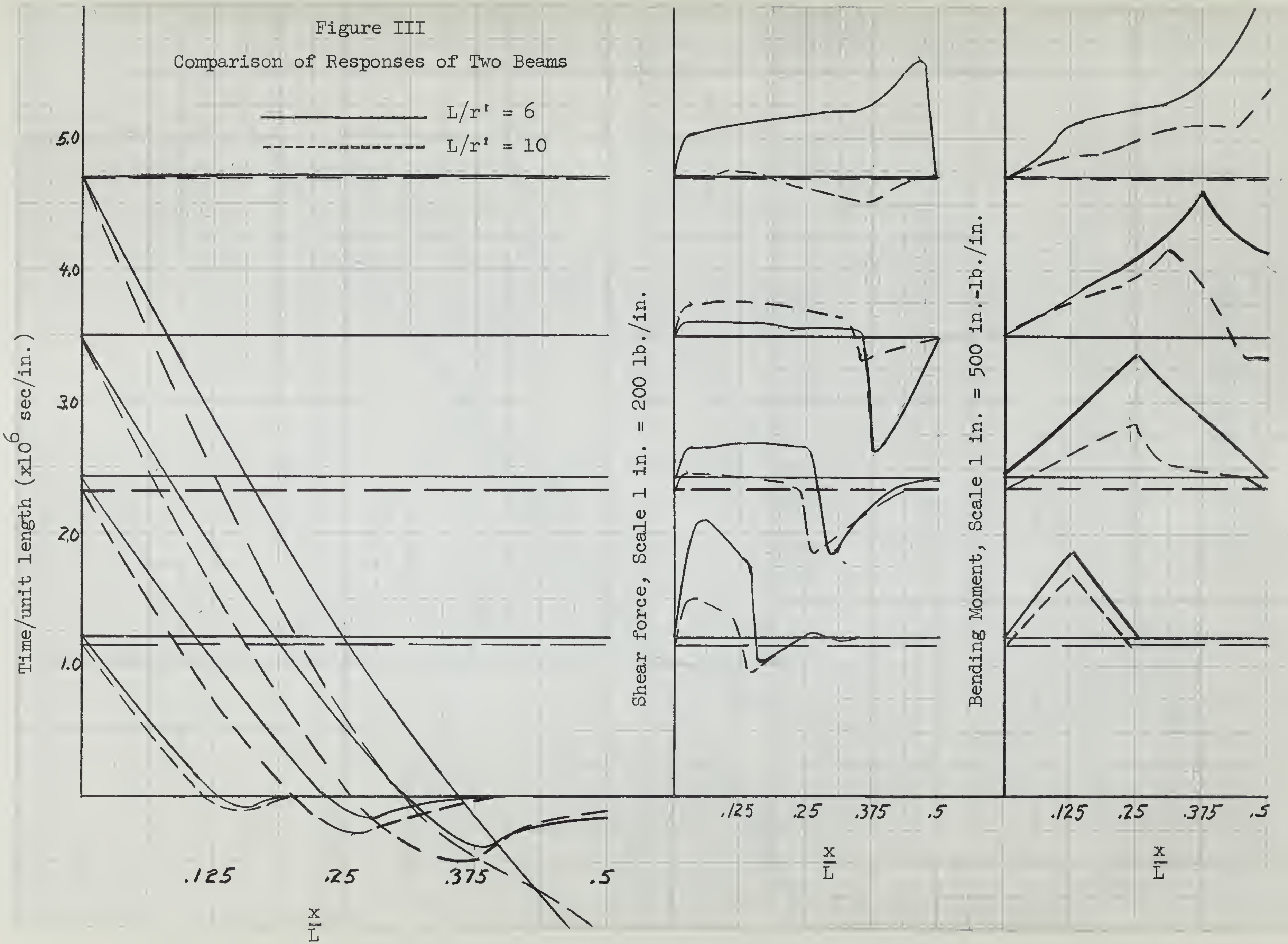
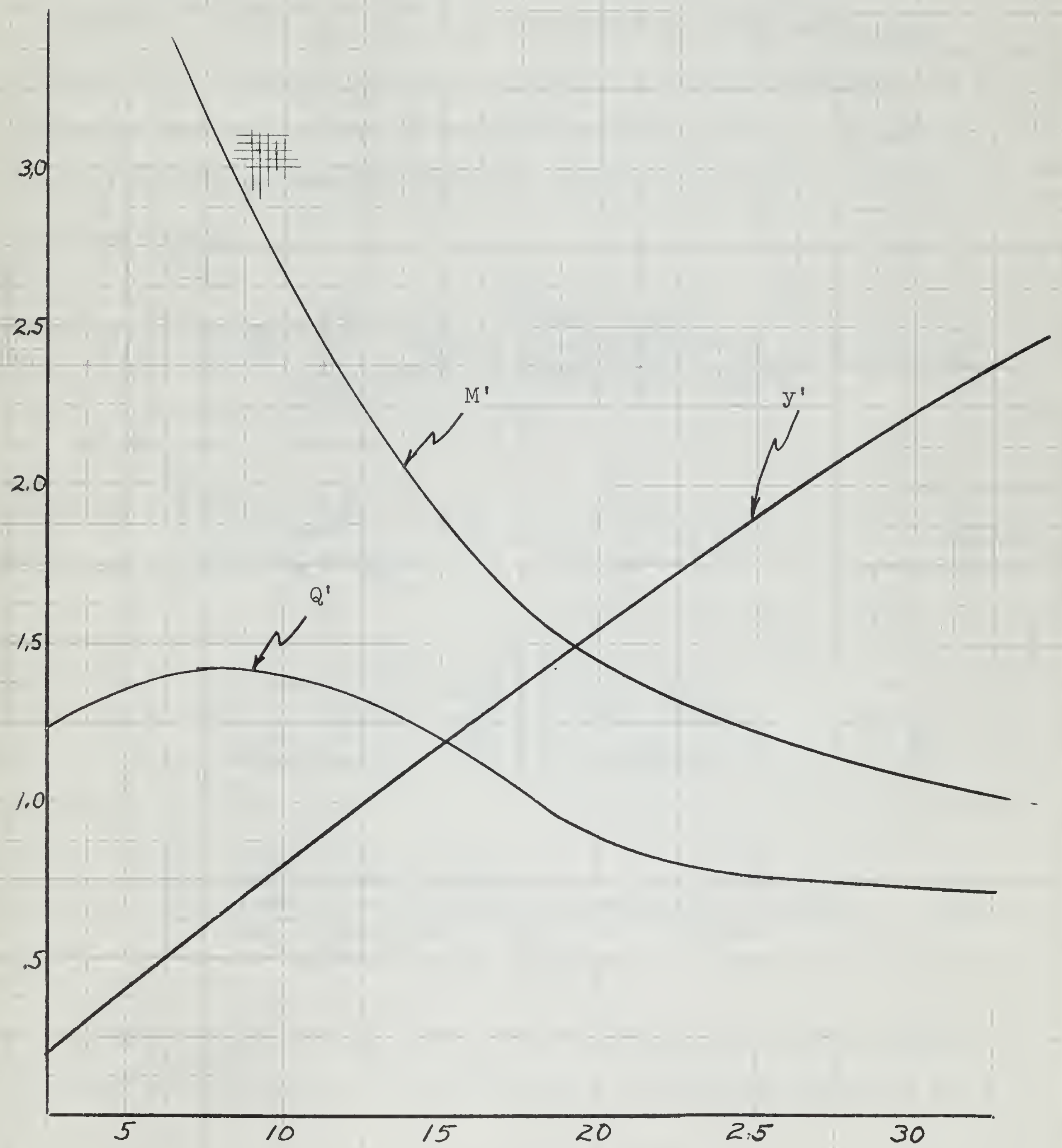


Figure III
Comparison of Responses of Two Beams

Figure IV

Non-Dimensional Maximum Deflection, Shear Force, and Bending Moment



IV. DISCUSSION OF RESULTS

Upon examination of Figure II, one's attention is immediately drawn to the negative motion that precedes the positive deflection as the disturbance travels into the beam. The curves for shear and bending moment tend to support this, but no physical reasoning is evident. To show how this occurs, look at the solutions for y_s and y_b . By putting equations (24) and (25) into (18) and (19), it is seen that, relative to the end points,

$$y_s = -\frac{4v}{\pi}\frac{m}{GA}\sum_{n=1,3,5\ldots}^{\infty}\frac{1}{nk_n^2}\left\{\left[\frac{GAk_n^2 - m\omega_{1n}^2}{m\omega_{2n}\,(\omega_{2n}^2 - \omega_{1n}^2)}\right]\omega_{2n}^2 \sin\omega_{2n}t\right.$$

$$\left. - \left[\frac{GAk_n^2 - m\omega_{2n}^2}{m\omega_{1n}\,(\omega_{2n}^2 - \omega_{1n}^2)}\right]\omega_{1n}^2 \sin\omega_{1n}t\right\}\sin k_n x \quad , \tag{49}$$

$$y_b = -\frac{4v}{\pi}\sum_{n=1,3,5\ldots}^{\infty}\frac{1}{n}\left\{\left[1 - \frac{m\omega_{2n}^2}{GAk_n^2}\right]\left[\frac{GAk_n^2 - m\omega_{1n}^2}{m\omega_{2n}\,(\omega_{2n}^2 - \omega_{1n}^2\omega}\right]\sin\omega_{2n}t\right.$$

$$\left. - \left[1 - \frac{m\omega_{1n}^2}{GAk_n^2}\right]\left[\frac{GAk_n^2 - m\omega_{2n}^2}{m\omega_{1n}\,(\omega_{2n}^2 - \omega_{1n}^2)}\right]\sin\omega_{1n}t\right\}\sin k_n x \quad . \tag{50}$$

Equations (49) and (50) show that the coefficients of each term in the series is the same as the coefficient of the corresponding term for y,

the net deflection, multiplied by a factor. For simplicity the nth term of equations (49) and (50) could be written,

$$y_{sn} = \frac{m\omega_{2n}^2}{GAk_n^2}\; y_{2n} - \frac{m\omega_{1n}^2}{GAk_n^2}\; y_{1n} \;, \tag{51}$$

and

$$y_{bn} = \left[1 - \frac{m\omega_{2n}^2}{GAk_n^2}\right] \cdot y_{2n} - \left[1 - \frac{m\omega_{1n}^2}{GAk_n^2}\right] \cdot y_{1n} \;, \tag{52}$$

where

y_{2n} = the nth sin $\omega_{2n}t$ term of y,

y_{1n} = the nth sin $\omega_{1n}t$ term of y.

As n goes to infinity,

$$y_{sn} \rightarrow \frac{E}{G}\, y_{2n} - y_{1n} \;,$$

$$y_{bn} \rightarrow \left[1 - \frac{E}{G}\right] y_{2n} - \left[1 - (1-)\right] y_{1n} \;,$$

where (1-) indicates that this factor approaches one. Now, by examining equation (26), the equation for y, it is seen that the coefficient of the sin $\omega_{1n}t$ term is negative. Furthermore, $\frac{E}{G}$ is always greater than one, e.g. for a solid circular shaft it is approximately 3.45. With these facts it can be seen that, for a circular shaft,

$$y_{sn} \rightarrow 3.45\, y_{2n} - y_{1n} > y_n \;, \tag{53}$$

$$y_{bn} \rightarrow -2.45\, y_{2n} \;. \tag{54}$$

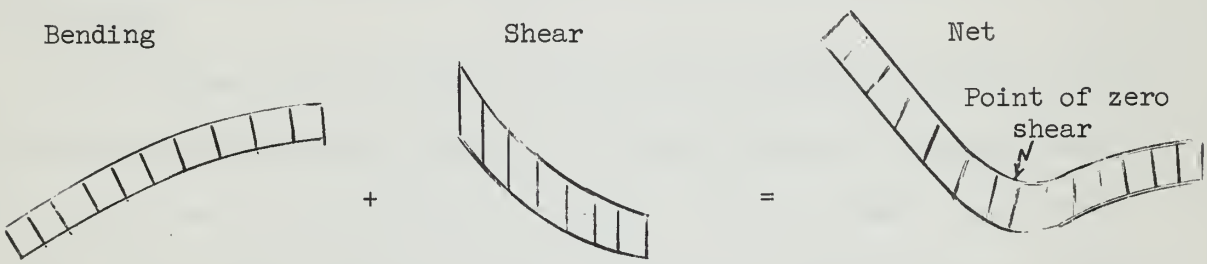
In other words, this shows that the deflection due to shear is greater than the net deflection and that the deflection due to bending is initially in the opposite direction! When it is recalled that the higher series of frequencies is associated with bending the the lower with shear, it follows that the bending effects should travel into the beam faster than the shear effects. The ratio of these speeds will be on the order of

$$\frac{\omega_{2n}}{\omega_{1n}} \approx \frac{E}{G} .$$

Utilizing these concepts, Figure V shows, in qualitative terms, how the resultant deflection occurs.

Figure V

Qualitative Description of Deflection Components



It can be deduced from this discussion, then, that the negative shear that precedes the expected positive values is due to the negative bending, and not the impact shear. This accounts for the fact that the negative portion keeps pace with the bending action instead of moving at the slower speed of the shear wave. Conversely, the rising, positive bending moment is due to the expected positive shear.

By equations (53) and (54), it can be seen that the energy contained in a beam is much greater than is indicated by the deflection alone. Even

though the deflections due to shear and bending are of opposite sign, their respective energies are additive. Thus the actual energy in a beam is much greater than classical theory predicts. Lack of time precludes deeper investigation of this point to gain quantitative information.

It was noticed that the curves for shear are relatively uneven and ragged. Even though the computations went to the forty-ninth mode, this is probably not enough since the series converges only as $\frac{1}{n}$, and as that only in the higher modes.

Upon examining Figure III, the lack of similarity in the two responses is noteworthy. This confirms that beams of two different length to radius of gyration ratios cannot be compared, even in a normalized form. Further, it is noted that, as predicted, the shear curves of the beam with the longer $\frac{L}{r'}$ look less like the ideal square wave than those of the shorter beam.

The accuracy of the curves in Figure IV is doubtful. Because of computer time limitations the search for the maxima may have been done with the time and position increments too large. For this reason also, the range of time examined may not have included the absolute maxima. However, the curves are included to show the general variation of the indicated quantities with $\frac{L}{r'}$.

The design method suggested in the Procedures section is dependent upon the availability of sufficient data from parent models. To obtain sufficient data to be generally useful, machine computation will probably be necessary. However, programming the equations derived in this report would not be too difficult. An example of how to utilize such derived curves is indicated in Appendix E.

A further refinement of the results of this work could be accomplished by incorporating them into elasticity theory to obtain the actual maximum stresses experienced in a beam. A calculation of this sort could be included in the computer program previously mentioned and a single family of curves of maximum stress versus $\frac{L}{r}$, for a range of $\frac{G}{E}$ would result.

Lastly, it is recognized that the results described herein should be confirmed by experiment before any actual engineering use of this is attempted. Such experiments are therefore recommended.

V. CONCLUSIONS

It is felt that the equations derived speak for themselves and require no further qualification. The equations show that the deflection due to shear predominates in the total deflection and that the deflection due to bending is in the opposite direction. The energy contained in a beam is greater than is indicated by the net deflection because of this fact. The equations imply that the disturbing effects of bending travel through the beam faster than those of shear.

It has been shown that beam response may be placed in a non-dimensional form whereby all beams with the same ratio of length to radius of gyration and the same ratio of shear modulus to modulus of elasticity, modified by a factor accounting for section shape, will have identical non-dimensional responses.

This fact forms the basis of a simplified method to obtain design calculations. The method is, however, dependent upon the compilation of response data from a number of parent models.

VI. RECOMMENDATIONS

It is recommended that the equations derived in this work be confirmed by experiment.

The indication that significantly greater energies are contained in a beam than is implied by the net deflection should be investigated further.

It is recommended that graphs for the non-dimensional form of the response of any beam be obtained. In doing so, care should be taken to assure that enough terms in the series be included to provide sufficient accuracy and to assure that enough of a span of time is examined, so that certainty of obtaining the maxima results.

Lastly, it is recommended that the results obtained in this work be applied to the theory of elasticity so that prediction of the magnitude, time and position of maximum stress could be calculated. It is recommended that the results of this be placed into a family of curves, where each curve is for a constant modified shear modulus to Young's modulus ratio, and the coordinates are maximum stress and the ratio of length to radius of gyration.

VII. APPENDICES

Appendix A. Application of Boundary Conditions

Equation (8) specifies that $X(0) = X(L) = 0$. However this could be specified more completely by,

$$X_s\,(0) = X_s\,(L) = X_b\,(0) = X_b\,(L) = 0\ .$$

Now let $C_1 = C_{1s} + C_{1b}$, $C_2 = C_{2s} + C_{2b}$, and so on. From this, one can look at the boundary conditions for the deflection due to bending separately. Thus,

$$X_b(x) = C_{1b}\sin kx + C_{2b}\cos kx + C_{3b}\sinh kx + C_{4b}\cosh kx\ , \qquad \text{(a)}$$

subject to boundary conditions

$$X_b\,(0) = X_b\,(L) = 0,$$

$$\left.\frac{\partial^2 X_b}{\partial x^2}\right|_{x=0} = \left.\frac{\partial^2 X_b}{\partial x^2}\right|_{x=L} = 0\ . \qquad \text{(b)}$$

Carrying out the indicated operations in (a) and (b) above,

$$X_b\,(0) = C_{2b} + C_{4b} = 0\ ,$$

$$\left.\frac{\partial^2 X_b}{\partial x^2}\right|_{x=0} = -C_{2b} + C_{4b} = 0\ .$$

Therefore $C_{2b} = C_{4b} = 0$.

Then

$$X_b\,(L) = C_{1b}\sin kL + C_{3b}\sinh kL = 0\ ,$$

$$\left.\frac{\partial^2 X_b}{\partial x^2}\right|_{x=L} = - C_{1b} \sin kL + C_{3b} \sin hkL = 0.$$

Since $\sin hkL$ can never be zero, the above can only be true if $C_{3b} = 0$ and $kL = n\pi$, $n = 1, 2, 3, \ldots$.

Thus,

$$X_b(x) = C_{1b} \sin \frac{n\pi}{L} x. \tag{c}$$

But the constants for X_s, the deflection due to shear, have not yet been satisfied. Thus,

$$X_s(x) = C_{1s} \sin kx + C_{2s} \cos kx + C_{3s} \sinh kx + C_{4s} \cosh kx , \tag{d}$$

and $$X_s(0) = C_{2s} + C_{4s} = 0 ,$$

$$X_s(L) = C_{1s} \sin kL + C_{2s} \cos kL + C_{3s} \sinh kL + C_{4s} \cosh kL = 0 .$$

But from (c) $k = \frac{n\pi}{L}$, thus,

$$X_s(L) = C_{2s} + C_{3s} \sinh n\pi + C_{4s} \cosh n\pi$$

$$= C_{2s} (1 - \cosh n\pi) + C_{3s} \sinh n\pi = 0 .$$

But this is only true if $C_{2s} = C_{3s} = 0$. Thus, $X = (C_{1s} + C_{1b}) \sin \frac{n\pi x}{L}$

$$= C \sin \frac{n\pi x}{L} .$$

Since the excitation of the beam motion is symmetrical, the slope of the beam at $x = \frac{L}{2}$ must be zero. This precludes any of the even numbered modes, and

$$X_n = C_n \sin \frac{n\pi x}{L}, \; n = 1,3,5,\ldots \tag{e}$$

Appendix B. More Convenient Form for Frequencies

Starting with equation (27), multiply numerator and denominator by,

$$\left[m + \left(\frac{EIm}{GA} + J\right) k_n^2\right] \mp \sqrt{\left[m + \left(\frac{EIm}{GA} + J\right) k_n^2\right]^2 - 4\,\frac{Jm}{GA}\,EIk_n^4}$$

resulting in,

$$\omega_n^2 = \frac{2\,EIk_n^4}{\left[m + \frac{EIm}{GA} + J\;k_n^2\right] \mp \sqrt{\left[m + \frac{EIm}{GA} + J\;k_n^2\right]^2 - 4\,\frac{JmEI}{GA}\,k_n^4}} \qquad \text{(a)}$$

Notice that,

$$\left[m + \left(\frac{EIm}{GA} + J\right) k_n^2\right] = \frac{EIm}{GA}\,k_n^2 \left[1 + \frac{mGA}{EIm\,k_n^2} + \frac{JGA}{EIm}\right].$$

But,

$$I = Ar'^2\,,\quad J = mr'^2\,,\quad \text{and} \quad k_n = \frac{n\pi}{L}\,.$$

Thus

$$\frac{EIm}{GA}\,k_n^2 \left[1 + \frac{GA}{EIk_n^2} + \frac{JGA}{EIm}\right] = \frac{EIm}{GA}\,k_n^2 \left[1 + \frac{G}{E}\left(1 + \left\{\frac{L}{\pi r'}\right\}^2 \frac{1}{n^2}\right)\right]$$

$$= \frac{EImk_n^2}{GA}\,z\,,$$

where

$$z = 1 + \frac{G}{E}\left(1 + \left\{\frac{L}{\pi r'}\right\}^2 \frac{1}{n^2}\right).$$

For the case under study, a solid circular beam, $\frac{L}{\pi r'} = \frac{2L}{\pi r}$.

Thus,

$$\omega_n^2 = \frac{2EIk_n^4}{\frac{EImk_n^2}{GA} z \pm \sqrt{\left(\frac{EImk_n^2}{GA} z\right)^2 - \frac{4mJEIk_n^4}{GA}}} .$$

Now multiply numerator and denominator of the last term under the radical by $\frac{EI}{GA}$ to obtain

$$\frac{4mJEIk_n^2}{GA} \left(\frac{EI}{GA}\right)\left(\frac{GA}{EI}\right) = \left(\frac{mEIk_n^2}{GA}\right)^2 . \, 4 \, . \, \frac{G}{E} .$$

Thus, by putting this into the equation for ω_n^2 , and factoring,

$$\omega_n^2 = \frac{2GAk_n^2}{m\left[z \pm \sqrt{z^2 - \frac{4G}{E}}\right]} \tag{b}$$

Appendix C. Proof of Convergance of Bending Moment Equation for Large Length to Radius Ratio.

From equations (29) and (30 a,b), it can be seen that, letting M_n be the nth term in the infinite series of M,

$$\lim_{\frac{L}{r'} \to \infty} M_n = \lim_{\frac{L}{r'} \to \infty} \frac{4v}{n\pi k_n} \left\{ \left[J \frac{m\omega_{2n}}{GAk_n} \right] \left[\frac{GAk_n^{\,2} - m\omega_{1n}^{\,2}}{m} \right] \sin \omega_{2n} t \right.$$

$$\left. + \left[\frac{m\omega_{1n}}{k_n} + J \left(k_n \omega_{1n} - \frac{m\omega_{1n}^{\,3}}{GAk_n} \right) \right] \left[\frac{m}{m} \right] \sin \omega_{1n} t \right\} \sin k_n x \quad .$$

Look at the expression from the above equation, $J \dfrac{m\omega_{2n}}{GAk_n}$.

Then

$$\lim_{\frac{L}{r'} \to \infty} \left(\frac{Jm\omega_{2n}}{GAk_n} \right) = \lim_{\frac{L}{r'} \to \infty} \left(\frac{mL}{GAk_n} \right) \left(\frac{J}{L} \quad \omega_{2n} \right) \quad .$$

With this and equation (27b), since $\dfrac{mL}{GAk_n}$ never goes to infinity,

$$\lim_{\frac{L}{r'} \to \infty} \left(\frac{Jm\omega_{2n}}{GAk_n} \right) = mL \sqrt{\frac{2m}{GA}} \quad \lim \left(\frac{\left(\frac{r'}{L}\right)^2}{z - \sqrt{z^2 - \frac{4G}{E}}} \right)^{\frac{1}{2}} \quad . \qquad \text{(a)}$$

Remembering that,

$$z = 1 + \frac{G}{E} + \frac{G}{E} \left(\frac{L}{n\pi r'} \right)^2 \quad ,$$

it can be seen that (a) goes to the form $\frac{0}{0}$ as $\frac{L}{r'}$ goes to infinity.

Clearly L' Hospital's rule applies here, but, in order to keep the application as simple as possible, recall that this calculation is only to show that this limit goes to zero. As the numerator is differentiated in accordance with L'Hospital's rule, it will always have zero as a limit as $\frac{L}{r'}$ goes to infinity. Thus the case will be proved simply by showing that the limit of some derivative of the denominator becomes other than zero.

To proceed, then,

$$\lim_{\frac{L}{r'} \to \infty} \frac{\partial}{\partial(\frac{L}{r'})} \left(z - \sqrt{z^2 - \frac{4G}{E}}\right)^{\frac{1}{2}} = \lim_{\frac{L}{r'} \to \infty} \frac{1}{2} \frac{\frac{G}{E}\left(\frac{1}{n\pi}\right)^2 \frac{L}{r'} - \frac{2z\frac{G}{E}\left(\frac{1}{n\pi}\right)^2 \frac{L}{r'}}{\sqrt{z^2 - \frac{4G}{E}}}}{\left(\cancel{\ } z - \sqrt{z^2 - \frac{4G}{E}}\right)^{\frac{1}{2}}} .$$

Since we are not interested in the actual magnitude of the limit, cancel out factors which are constant with respect to $\frac{L}{r'}$. Thus we get

$$\lim_{\frac{L}{r'} \to \infty} \left[\frac{\frac{L}{r'}\left(1 - \frac{2z}{\sqrt{z^2 - \frac{4G}{E}}}\right)}{\left(z - \sqrt{z^2 - \frac{4G}{E}}\right)^{\frac{1}{2}}}\right] = \frac{\infty}{0} = \infty$$

Therefore the sin $\omega_{2n}t$ term goes to zero in the limit as $\frac{L}{r'}$ goes to infinity.

By similar reasoning, simply by multiplying J by $\frac{L}{L}$ in the sin $\omega_{1n}t$ term,

$$\lim_{\frac{L}{r'} \to \infty} \left[\frac{m\omega_{1n}}{k_n} - J\frac{L}{L}\left(k_n \omega_{1n} - \frac{m\omega_{1n}^3}{GAk_n}\right)\right] = \frac{m\omega_{1n}}{k_n} \qquad (c)$$

Then,

$$\lim_{\frac{L}{r'} \to \infty} M_n = \frac{4v}{n\pi k_n} \left(\frac{m\omega_{1n}}{k_n} \right) \sin \omega_{1n} t \sin k_n x \quad .$$

Appendix D. Derivation of Damping Factor

A beam experiencing transverse motion could be interpreted as an extremely complex spring-mass system. Since the energy of any spring-mass system is proportional to the square of its deflection, one could write

$$E_i \propto y_i^2 \quad ,$$

where

E_i = Energy of the ith cycle,

y_i = Deflection of the ith cycle.

If the energy lost per cycle is two per cent of the energy of that cycle, then

$$\frac{E_i - E_{i+1}}{E_i} = \frac{y_i^2 - y_{i+1}^2}{y_i^2} = 1 - \frac{y_{i+1}^2}{Y_i^2} = 0.02 \ . \qquad \text{(a)}$$

From elementary vibration theory the damping factor is of the form,

$$\text{Damping factor} = e^{-\omega_n B t} \quad , \qquad \text{(b)}$$

where

B = fraction of critical damping,

and the frequency, modified by damping is

$$\omega_d = \omega_n \sqrt{1 - B^2} \ . \qquad \text{(c)}$$

From (c) the time for one cycle, or the period, is

$$T_d = \frac{2\pi}{\omega_n \sqrt{1 - B^2}} \ . \qquad \text{(d)}$$

Clearly, then, from (b) and (d) ,

$$\frac{y_{i+1}}{y_i} = e^{-\omega_n B T_d} = e^{\frac{-2\pi B}{\sqrt{1-B^2}}} .$$

If B is small compared to 1,

$$\frac{y_{i+1}}{y_i} \approx e^{-2\pi B} . \tag{e}$$

Combining (a) and (e) one obtains

$$1 - \frac{y_{i+1}^2}{y_i^2} = 1 - e^{-4\pi B} = .02 .$$

Solving this for B one obtains

$$B = 0.00161 . \tag{f}$$

Notice that the effect of B on the natural frequency in (c) above is small. Therefore this effect shall be neglected.

If each term in the equations for y, y_s, and y_b is multiplied by its appropriate damping factor and the equations for Q and M are derived again, it can be seen that, since $B << 1$, the same equations result approximately except that each term is multiplied by the appropriate damping factor.

Appendix E. Example Problem

Assume we have a solid steel circular shaft 18 inches in diameter. What would be the maximum length that could be allowed between supports if the beam is not to deflect more than .1 inch when subjected to a shock velocity of 35 inches per second?

Let steel have the characteristics,

$$\rho = 7.33 \times 10^{-4} \ \frac{\text{lb.-sec.}^2}{\text{in.}^4}$$

$$G' = 11.2 \times 10^6 \text{ psi} .$$

It can be seen that, by equation (47),

$$k' = \frac{br'^2}{\int_0^C hdA} = \frac{R \times R^2}{4\int_0^R 2h\sqrt{R^2-h^2}\, dh} = \frac{3}{4} .$$

Thus

$$G = \frac{3}{4} G' = 8.45 \times 10^6 \text{ psi.}$$

By equation (44),

$$y' = \frac{y}{vL\sqrt{\frac{m}{GA}}} = \frac{y}{Lv}\sqrt{\frac{G}{\rho}} = \frac{.1}{L \times 35}\sqrt{\frac{8.45 \times 10^6}{7.33 \times 10^{-4}}} = \frac{308}{L} .$$

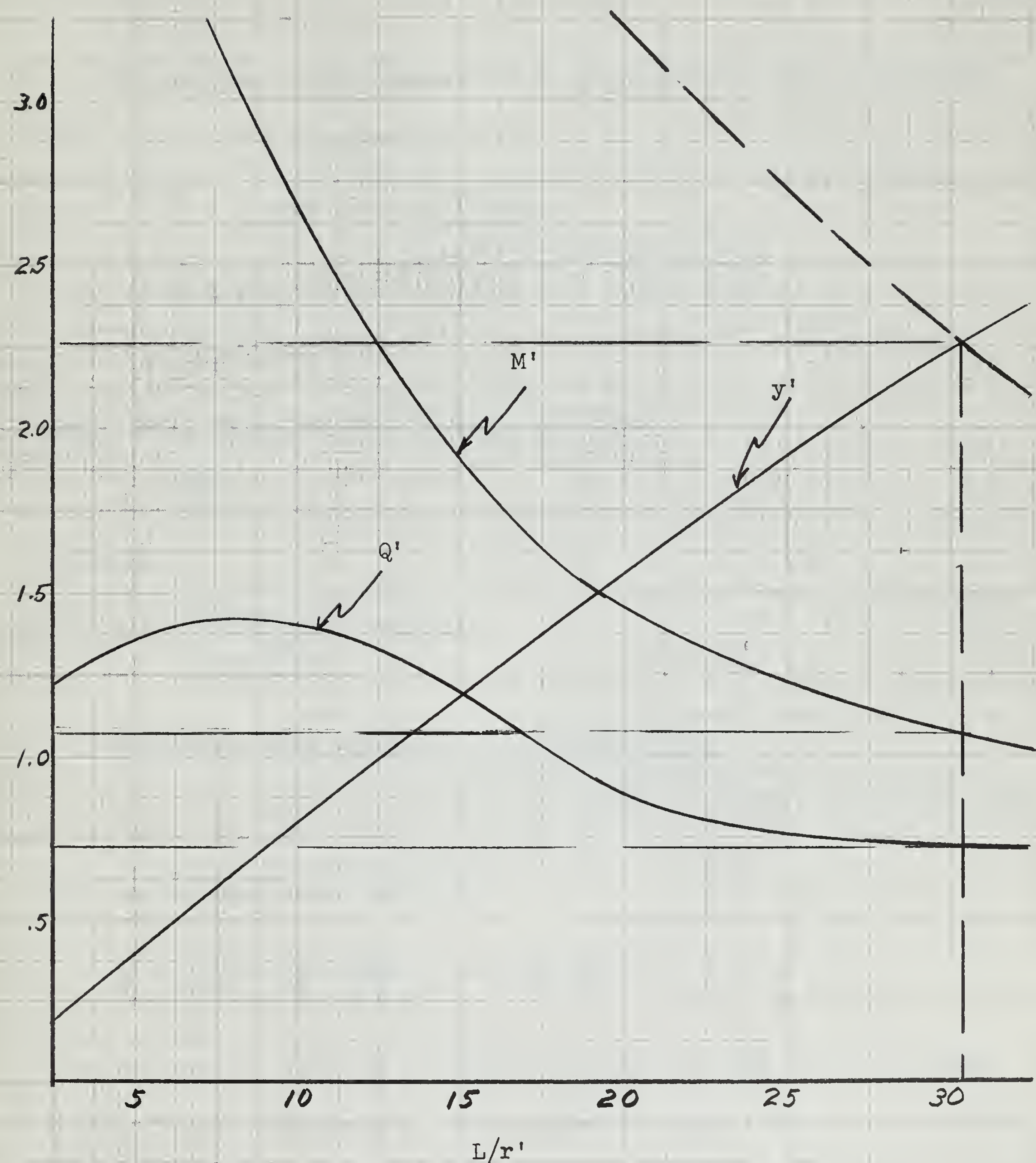
For a circular cross section, $r' = \frac{R}{2}$.

Tabulate y' and $\frac{L}{r'}$, for several values of L, as indicated below:

L (inches)	y' $(\frac{308}{L})$	$\frac{L}{r'}$, $(\frac{L}{4.5})$
96	3.1	21.3
120	2.56	26.7
144	2.13	32.0

Figure VI

Determination of Maximum Length of Beam According to Example Problem



Plotting these values in Figure VI, which is just a copy of Figure IV, it can be seen that the required L/r' is 30.3. Then

$$L = \frac{L}{r'} \times r' = 30.3 \times 4.5 = 136 \text{ inches.}$$

The maximum bending moment for a beam with this L/r' is obtained from Figure VI and equation 46, i.e.,

$$M = M' \left(vL\sqrt{mGA}\right) = M' \left(vLA\sqrt{\rho G}\right)$$

$$= 35 \times 136 \times \pi \times 9^2 \sqrt{7.33 \times 10^{-4} \times 8.45 \times 10^6}$$

$$= 419{,}000 \text{ in.-lbs.}$$

Using the formula for maximum bending stress,

$$S_b = \frac{MC}{I},$$

we find

$$S_b = \frac{419{,}000 \times 9}{\left(\frac{\pi \times 9^4}{4}\right)} = 723 \text{ psi.}$$

Similarly, from equation (45) and Figure VI,

$$Q = Q' \left(v\sqrt{mGA}\right) = .73 \times 35 \times \pi \times 9^2 \sqrt{7.33 \times 10^{-4} \times 8.45 \times 10^6} = 2010 \text{ lb.}$$

Now, the maximum shear stress is

$$S_s = \frac{1}{k'}\,\frac{Q}{A} = \frac{4}{3} \times \frac{2010}{\pi \times 9^2} = 10.52 \text{ psi}.$$

Clearly the deflection governs in this case, but if the allowable deflection had been greater, the acceptable length would have increased. This would cause a rise in bending stress. A point would thus be reached where the stresses govern the length.

Appendix F. References

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